

# The Laws of Linear Combination

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# Goals for this Module

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- # In this module, we cover
    - What is a linear combination? Basic definitions and terminology
    - Key aspects of the behavior of linear combinations
      - The Mean of a linear combination
      - The Variance of a linear combination
      - The Covariance between two linear combinations
      - The Correlation between two linear combinations
    - Applications
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# What is a linear combination?

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## # A classic example

- Consider a course, in which there are two exams, a midterm and a final.
- The exams are not “weighted equally.” The final exam “counts twice as much” as the midterm.

# Course Grades

# The first few rows of the data matrix for our hypothetical class might look like this:

Person	Midterm	Final	Course
A	90	78	82
B	82	88	86
C	74	86	82
D	90	96	94

# Course Grades

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- # The course grade was produced with the following equation

$$G_i = \frac{1}{3}X_i + \frac{2}{3}Y_i$$

where  $G$  is the course grade,  $X$  is the midterm grade, and  $Y$  is the final exam grade.

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# Course Grades

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# We say that the variables  $G$ ,  $X$ , and  $Y$  follow the linear combination rule  $G = (1/3)X + (2/3)Y$  and that  $G$  is a *linear combination of  $X$  and  $Y$* .

# More generally, any expression of the form

$$C = aX + bY$$

is a linear combination of  $X$  and  $Y$ . The constants  $a$  and  $b$  in the above expression are called *linear weights*.

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# Specifying a Linear Combination

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- # Any linear combination of two variables is, in a sense, defined by its linear weights. Suppose we have two lists of numbers,  $X$  and  $Y$ . Below is a table of some common linear combinations:

# Specifying a Linear Combination

$X$	$Y$	<i>Name of LC</i>
+1	+1	Sum
+1	-1	Difference
+(1/2)	+(1/2)	Average



# Learning to “read” a LC

- # It is important to be able to examine an expression and determine the following:
  - Is the expression a LC?
  - What are the linear weights?
- # Example: Is this a linear combination of  $\bar{X}_{\bullet 1}$  and  $\bar{X}_{\bullet 2}$ ?

$$\bar{X}_{\bullet\bullet} = \frac{N_1 \bar{X}_{\bullet 1} + N_2 \bar{X}_{\bullet 2}}{N_1 + N_2}$$

# Learning to “read” a L.C.

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# What are the linear weights?

$$\bar{X}_{..} = \frac{N_1 \bar{X}_{.1} + N_2 \bar{X}_{.2}}{N_1 + N_2}$$

# Learning to “read” a L.C.

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# To answer the above, you must re-express  $\bar{X}_{..}$  in the form

$$\bar{X}_{..} = a\bar{X}_{.1} + b\bar{X}_{.2}$$



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# Consider

$$\bar{X}_{\bullet} = \frac{1}{N} \sum_{i=1}^N X_i$$

# Is  $\bar{X}_{\bullet}$  a linear combination of the  $X_i$  ?

# If so, what are the linear weights?

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# The Mean of a Linear Combination

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- # Return to our original example. Suppose now that this represents *all* the data --- the class is composed of only 4 people.

# The Mean of a Linear Combination

Person	Midterm ( $X$ )	Final ( $Y$ )	Course ( $G$ )
A	90	78	82
B	82	88	86
C	74	86	82
D	90	96	94
Mean	84	87	86

The grades were produced with the formula  $G = (1/3)X + (2/3)Y$ . Notice that the group means follow the same rule!

# The Linear Combination Rule for Means

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# If  $W_i = aX_i + bY_i$  then

$$\bar{W}_\bullet = a\bar{X}_\bullet + b\bar{Y}_\bullet$$

# In the course grades,  $X$  has a mean of 84 and  $Y$  has a mean of 87. So the mean of the final grades must be  $(1/3) 84 + (2/3)87 = 86$

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# Proving the Linear Combination Rule for Means

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- # Proving the preceding rule is a straightforward exercise in summation algebra. (C.P.)



# The Variance of a Linear Combination

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- # Although the rule for the mean of a LC is simple, the rule for the variance is not so simple. Proving this rule is trivial with matrix algebra, but much more challenging with summation algebra. (See the “Key Concepts” handout for a detailed proof.)
  - # At this stage, we will present this rule as a “heuristic rule,” a procedure that is easy and yields the correct answer.
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# The Variance of a Linear Combination – A Heuristic Rule

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- # Write the linear combination as a rule for variables. For example, if the LC simply sums the  $X$  and  $Y$  scores, write

$$X + Y$$

- # Algebraically square the above expression

$$(X + Y)^2 = X^2 + Y^2 + 2XY$$

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# The Variance of a Linear Combination – A Heuristic Rule

- # Take the resulting expression

$$X^2 + Y^2 + 2XY$$

- # Apply a “conversion rule”. (1) Constants are left unchanged. (2) Squared variables become the variance of the variable. (3) Products of two variables become the covariance of the two variables

$$S_x^2 + S_y^2 + 2S_{xy}$$

# The Heuristic Rule – An Example

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- # Develop an expression for the variance of the following linear combination

$$X - 2Y$$

# The Heuristic Rule – An Example

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$$(X - 2Y)^2 = X^2 + 4Y^2 - 4XY$$

So

$$S_{X-2Y}^2 = S_X^2 + 4S_Y^2 - 4S_{XY}$$

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# Two Linear Combinations on the Same Data

- # It is quite common to have more than one linear combination on the same columns of numbers

$X$	$Y$	$(X + Y)$	$(X - Y)$
1	3	4	-2
2	1	3	1
3	2	5	1

# Two Linear Combinations on the Same Data

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Can you think of a common situation in psychology where more than one linear combination is computed on the same data?

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# Covariance of Two Linear Combinations – A Heuristic Rule

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- # Consider two linear combinations on the same data.
  - # To compute their covariance:
    - write the algebraic product of the two LC expressions, then
    - apply the conversion rule
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# Covariance of Two Linear Combinations

## # Example

$$L = X + Y$$

$$M = X - Y$$

$$S_{LM} = S_{X+Y, X-Y} = ?$$

# Covariance of Two Linear Combinations

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$$(X + Y)(X - Y) = X^2 - Y^2$$

Hence

$$S_{LM} = S_X^2 - S_Y^2$$

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# An SPSS Example

- # Start up SPSS and create a file with these data.

$X$	$Y$
1	4
2	3
3	5
4	2
5	1

# An SPSS Example

# Next, we will compute the linear combinations  $X + Y$  and  $X - Y$  in the variables labeled  $L$  and  $M$ .

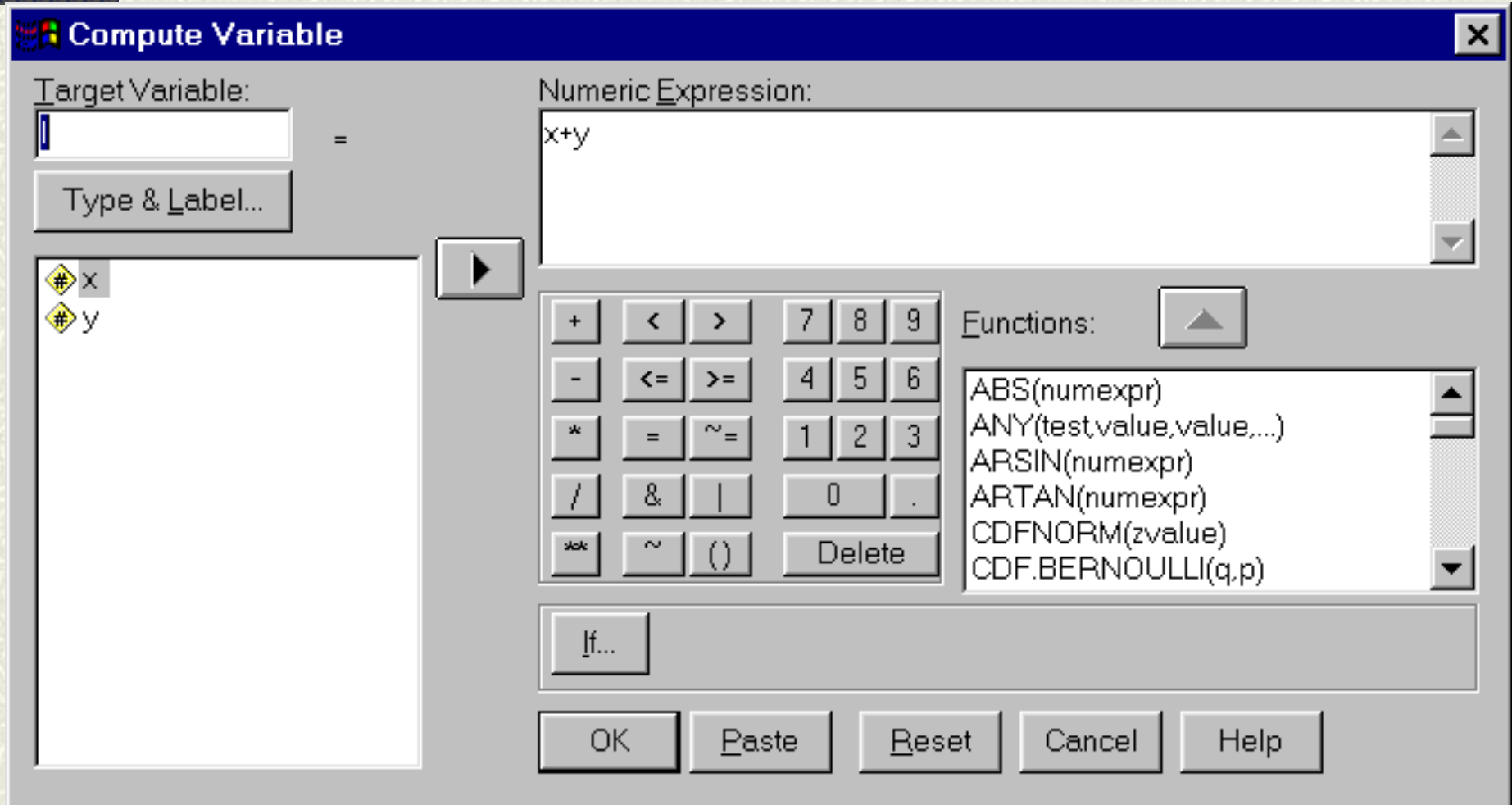
$X$	$Y$	$L$	$M$
1	4	5	-3
2	3	5	-1
3	5	8	-2
4	2	6	2
5	1	6	4

# An SPSS Example

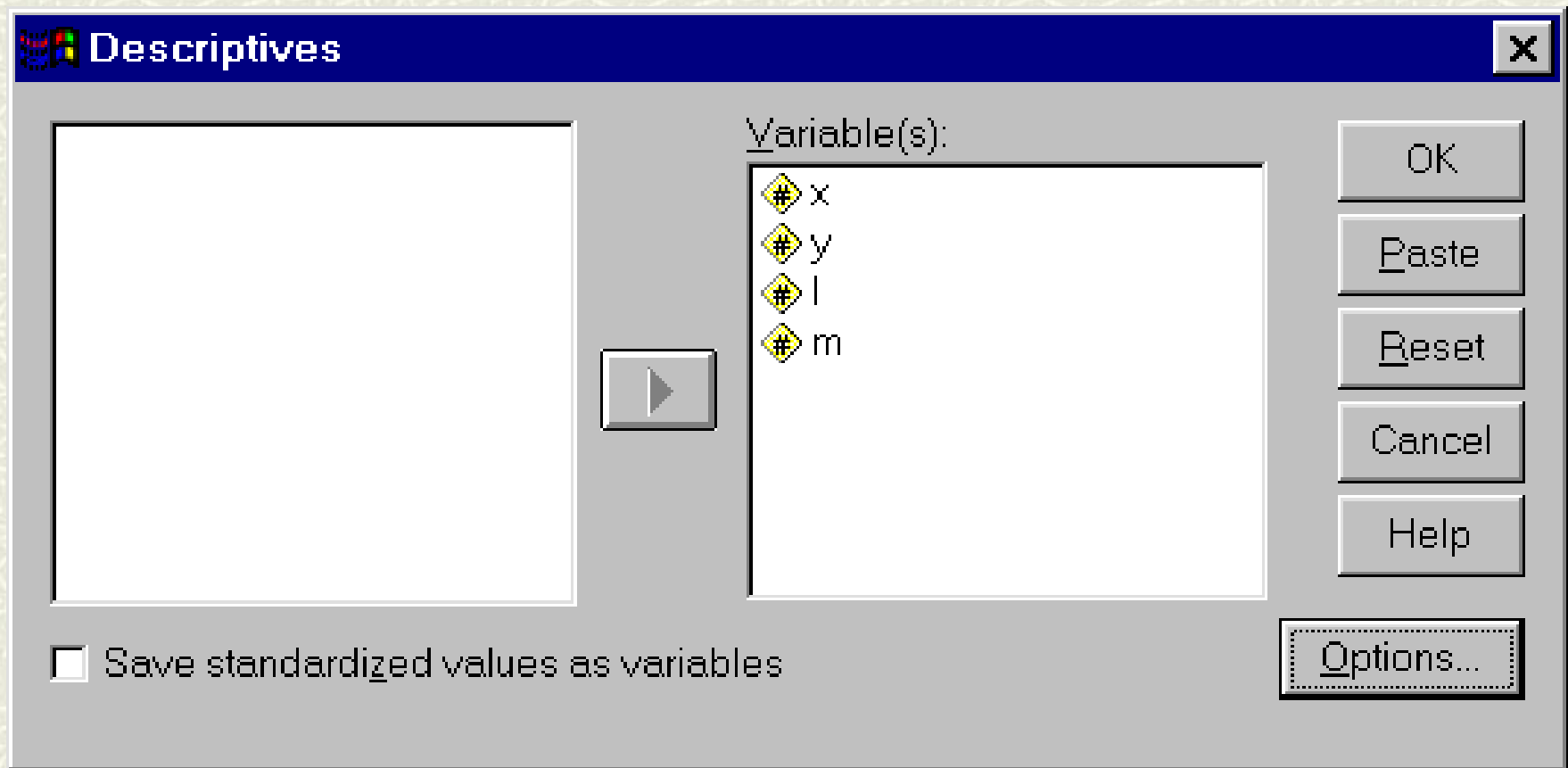
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- # Using SPSS, we will next compute descriptive statistics for the variables, and verify that they in fact obey the laws of linear combinations.

# An SPSS Example



# An SPSS Example



# An SPSS Example

**Descriptives: Options** [X]

Mean       Sum

Dispersion

Std. deviation       Minimum

Variance       Maximum

Range       S.E. mean

Distribution

Kurtosis       Skewness

Display Order

Variable list

Alphabetic

Ascending means

Descending means

Continue

Cancel

Help

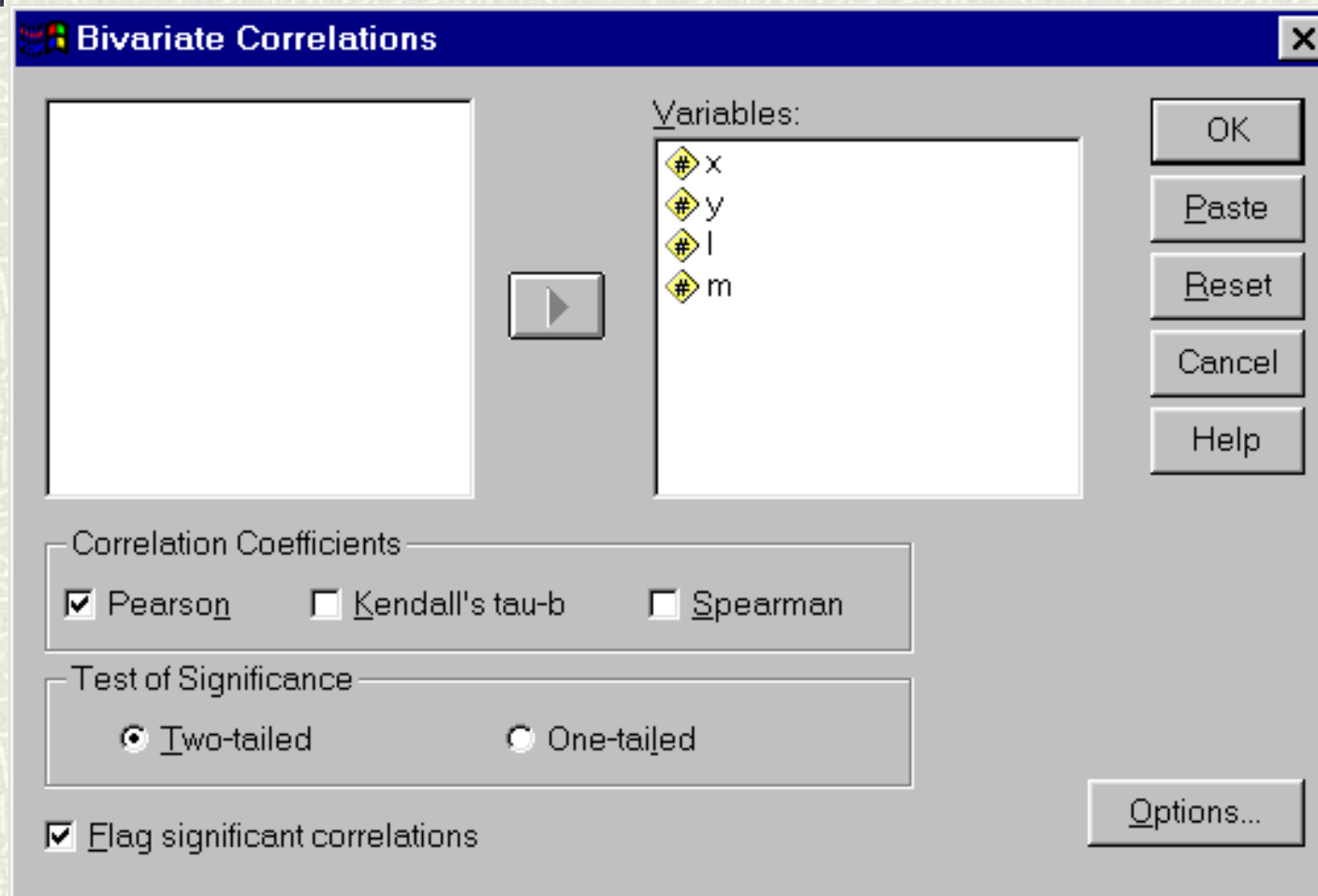


# An SPSS Example

## Descriptive Statistics

	N	Min	Max	Mean	Std. Dev.	Variance
X	5	1	5	3.00	1.581	2.500
Y	5	1	5	3.00	1.581	2.500
L	5	5.00	8.00	6.0000	1.22474	1.500
M	5	-3.00	4.00	.0000	2.91548	8.500

# Computing Covariances



# Computing Covariances

**Bivariate Correlations: Options** [X]

Statistics

- Means and standard deviations
- Cross-product deviations and covariances

Missing Values

- Exclude cases pairwise
- Exclude cases listwise

Continue

Cancel

Help

# An SPSS Example

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$$S_Y^2 = S_X^2 = 2.5 \quad S_{YX} = -1.75$$

$$S_L^2 = 1.5 \quad S_M^2 = 8.5$$

$$S_{LM} = 0$$

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# Comment

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- # Note how we have learned a simple way to create two lists of numbers with a precisely zero covariance (and correlation).

# The “Rich Get Richer” Phenomenon

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- # We have discussed the overall benefits of scaling exam grades to a common metric.
  - # A fair number of students, converting their scores to  $Z$  scores following each exam, find their final grades disappointing, and in fact suspect that there has been some error in computation. We use the theory of linear combinations to develop an understanding of why this might happen.
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# The “Rich Get Richer” Phenomenon

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- # Suppose the grades from exam 1 and exam 2 are labeled  $X$  and  $Y$ , and that  $r_{XY} = 0.5$
  - # Suppose further that the scores are in  $Z$  score form, so that the means are 0 and the standard deviations are 1 on each exam.
  - # The professor computes a final grade by *averaging the grades on the two exams, then rescaling these averaged grades so that they have a mean of 0 and a standard deviation of 1.*
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# The “Rich Get Richer” Phenomenon

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- # Under the conditions just described, suppose a student has a  $Z$  score of  $-1$  on both exams. What will the student’s final grade be?
  - # To answer this question, we have to examine the implications of the professor’s grading method carefully.
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# The “Rich Get Richer” Phenomenon

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- # By averaging the two exams, the professor used the linear combination formula

$$G = \frac{1}{2}X + \frac{1}{2}Y$$

- # If  $X$  and  $Y$  are in  $Z$  score form, find the mean and variance of  $G$ , the final grades.
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# The “Rich Get Richer” Phenomenon

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# Using the linear combination rule for means, we find that

$$\bar{G}_{\bullet} = \frac{1}{2} \bar{X}_{\bullet} + \frac{1}{2} \bar{Y}_{\bullet} = 0$$

# The “Rich Get Richer” Phenomenon

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- # Using the heuristic rule for variances, we find that

$$S_G^2 = \frac{1}{4}S_X^2 + \frac{1}{4}S_Y^2 + \frac{1}{2}S_{XY}$$

- # However, since  $X$  and  $Y$  are in  $Z$  score form, the variances are both 1, and the covariance is equal to the correlation
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# The “Rich Get Richer” Phenomenon

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# So

$$S_G^2 = \frac{1}{4}(1) + \frac{1}{4}(1) + \frac{1}{2}(.5) = .75$$

and

$$S_G = \sqrt{.75} = .866$$

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# The “Rich Get Richer” Phenomenon

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The scores are no longer in  $Z$  score form, because their mean is zero, but their standard deviation is not 1.

What must the professor do to put the scores back into  $Z$  score form?

*Why will this cause substantial disappointment for a student who had  $Z$  scores of  $-1$  on both exams?*

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